DATE OF EXAM: 26.02.2016
SUBJECT NAME: Functional Analysis - MIDTERM Exam 2015-16 - M. Math First Year

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1. **Problem:** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Prove that for each $x \in \mathcal{H}$

$$\lim_{m\to\infty}\langle x, e_m\rangle = 0.$$

Solution: Since $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$, for every $x \in \mathcal{H}$ from Parseval's Identity we have:

$$\sum_{m=1}^{\infty} |\langle x, e_m\rangle|^2 = \| x \|^2 < \infty.$$

Hence

$$\lim_{m\to\infty}\langle x, e_m\rangle = 0.$$

□

2. **Problem:** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be a pair of equivalent norms on the linear space X. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ yields equivalent norms on the set of bounded linear operators on X.

Solution: Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on the linear space X, there exists constants $\alpha_1, \beta_1 > 0$ such that $\alpha_1 \| x \|_1 \leq \| x \|_2 \leq \beta_1 \| x \|_1$. Let T be a bounded linear operator on X. Also there exists constants $\alpha_2, \beta_2 > 0$ such that $\alpha_2 \| Tx \|_1 \leq \| Tx \|_2 \leq \beta_2 \| Tx \|_1$. Thus we have:

$$\alpha_2 \| Tx \|_1 \leq \| Tx \|_2 \leq \| T \|_2 \| x \|_2 \leq \| T \|_2 \, \beta_1 \| x \|_1 .$$

Hence $\| T \|_1 \leq \frac{\beta_1}{\alpha_2} \| T \|_2$. Interchanging the role of $\|\cdot\|_1$ and $\|\cdot\|_2$, we get the other direction.

□

3. **Problem:** Let X and Y be two Banach spaces over $\mathbb{C}$ and $\phi : X \times Y \to \mathbb{C}$ be a bilinear map and continuous in each variable separately. Prove that ϕ is jointly continuous.

Solution: Let $\phi_x : Y \to \mathbb{C}$ and $\phi^y : X \to \mathbb{C}$ is defined by $\phi_x(y) = \phi(x,y) = \phi^y(x)$. Let $E = \{y \in Y : \| y \| \leq 1\}$. Now

$$\| \phi^y(x) \| = \| \phi(x,y) = \| \phi_x(y) \| \leq \| \phi_x \| \| y \| \leq \| \phi_x \| .$$

So, $\{\| \phi^y(x) \|\}$ is bounded. Thus $\{\| \phi^y \| : y \in E\}$ is bounded by UBP. Let $\| \phi^y \| \leq \alpha$. Let $x \in X, y \in Y$. Take $z = \frac{y}{\|y\|}$. Then $z \in E$. So, $\| \phi(x,z) \| = \| \phi^z(x) \| \leq \| \phi^z \| \| x \| \leq \alpha \| x \|$. Now note that $\| \phi(x,z) \| = \frac{1}{\|y\|} \| \phi(x,y) \|$. So $\| \phi(x,y) \| = \| y \| \| \phi(x,z) \| \leq \alpha \| x \| \| y \|$. To check the joint continuity of ϕ, let $(x_n, y_n) \to (x,y)$, i.e. $x_n \to x, y_n \to y$. Note that $\phi(x_n,y_n) - \phi(x,y_n) = \phi(x_n - x, y_n)$ and $\phi(x,y_n) - \phi(x,y) = \phi(x, y_n - y)$. Hence $\| \phi(x_n,y_n) - \phi(x,y) \| \leq \| \phi(x_n - x, y_n) \| + \| \phi(x, y_n - y) \| \leq \alpha \| x_n - x \| \| y_n \| + \alpha \| x \| \| y_n - y \| \to 0$.

□

4. **Problem:** Let $\mathcal{F}$ be a proper finite dimensional subspace of a normed linear space X. Prove that there exists an unit vector $x \in X$ such that

$$\| x - y \| \geq 1 \quad (\forall y \imath \mathcal{F}).$$

Solution: Choose $x \in X$ with $x \notin \mathcal{F}$. Let $d = d(x, \mathcal{F})$. Let $d = d(x, \mathcal{F}) = \inf\{\| x - y \|: \quad y \in \mathcal{F}\}$. Let, $r = \| x \| + d + 1$, and $S = \{y \in Y : \| y \| \leq r\}$. Then S is a closed and bounded subset of the finite dimensional space $\mathcal{F}$ and hence is compact. The function $\phi : \mathcal{F} \to \mathbb{R}$ defined by

$$\phi(y) = \| y - x \|,$$

is real and continuous on $\mathcal{F}$. Hence there is a $y_0 \in S$ such that $\phi(y_0) \leq \phi(y)$; for all $y \in S$. Again there is $y_1 \in \mathcal{F}$ such that $\| x - y_1 \| < d + 1$. Then, $\| y_1 \| \leq \| y_1 - x \| + \| x \| < d + 1 + \| x \| = r$. Hence $y_1 \in S$ and so

$$\phi(y_0) \leq \phi(y_1) = \| y_1 - x \| < d + 1.$$

Let $y \in \mathcal{F}$. If $y \notin S$, then

$$\| x \| + d + 1 = r < \| y \| \leq \| y - x \| + \| x \| .$$

Hence in this case we get

$$\phi(y_0) < d + 1 < \| y - x \| .$$

On the other hand if $y \in S$, then

$$\phi(y_0) \leq \phi(y) = \| y - x \| .$$

Thus $\| y_0 - x \| = \inf\{\| x - y \|: \quad y \in \mathcal{F}\} = d$. If $d = 0$, then $x = y_0 \in Y$, which is not true. Therefore $d > 0$. Let $x_1 = \frac{x - y_0}{d}$, then $\| x_1 \| = 1$ and for all $y \in \mathcal{F}$,

$$\| x_1 - y \| = \| \frac{1}{d}(x - z) \| \geq 1,$$

since $z = y_0 + dy \in Y$. This also implies that $d(x_1, \mathcal{F}) = 1$, since $\| x_1 - 0 \| = 1$ and $0 \in Y$.

□

5. **Problem:** Prove or disprove (with justification): $T : c_{00} \to \mathbb{C}$ defined by $T(\{a_n\}) = \sum_n a_n$ is a bounded linear functional (w.r.t. usual sup norm).

Solution: Consider $x_n \in c_{00}$ defined by

$$\begin{aligned} x_n(j) &= 1 \quad \text{for} \quad 1 \leq j \leq n, \\ &= 0 \quad \text{for} \quad j > n. \end{aligned}$$

Then $\| x_n \|_\infty = 1$, and $T(\{x_n\}) = n$. If T is continuous then there exists $\alpha > 0$ such that

$$|T(\{x_n\})| \leq \alpha \| x_n \|_\infty = \alpha,$$

that is, $n \leq \alpha$, for all n. The above implies that T is not continuous.

□

6. **Problem:** Let T be a linear map on a Hilbert space $\mathcal{H}$ and

$$\langle Tx, y \rangle = \langle x, Ty \rangle,$$

for all $x, y \in \mathcal{H}$. Prove that T is bounded.

Solution: From a special case of the uniform boundedness principle we have: If $\mathcal{H}$ is a Hilbert space, $E \subset \mathcal{H}$, and for every $y \in \mathcal{H}$; there exists a constant $\alpha(y) > 0$, such that

$$|\langle x, y\rangle| \leq \alpha(y); \quad \forall x \in E,$$

then $\sup\{\| x \| : x \in E\} < \infty$. Now, take $E = \{Tx : \| x \| = 1\}$ and observe that for every $y \in \mathcal{H}$

$$|\langle Tx, y\rangle| = |\langle x, Ty\rangle| \leq \| x \| \| Ty \| \leq \| Ty \| .$$

Hence by the first statement T is bounded.

$\square$

7. **Problem:** Let $\mathcal{S}$ be a subspace of a Hilbert space $\mathcal{H}$. Prove that

$$(\mathcal{S}^{\perp})^{\perp} = \overline{\mathcal{S}}.$$

Solution: Let $Y = \overline{\mathcal{S}}$. First we shall show that $\mathcal{S}^{\perp} = Y^{\perp}$. Since $\mathcal{S} \subset Y$, we have $Y^{\perp} \subset \mathcal{S}^{\perp}$. Conversely let, $x \in \mathcal{S}^{\perp}$. We need to show that $x \in Y^{\perp}$. Take any $y \in Y$, then there exists $\{z_n\} \in \mathcal{S}$ such that $z_n \to y$. So $\langle x, z_n\rangle \to \langle x, y\rangle$. But $\langle x, z_n\rangle = 0$ and hence $\langle x, y\rangle = 0$. Thus $\mathcal{S}^{\perp} \subset Y^{\perp}$ and hence $\mathcal{S}^{\perp} = Y^{\perp}$. Since Y is a closed subspace of a Hilbert space; so Y is complete. Hence $(Y^{\perp})^{\perp} = Y$. As a result we have

$$(\mathcal{S}^{\perp})^{\perp} = \overline{\mathcal{S}}.$$

$\square$

8. **Problem:** Let A be a $\sigma-$ finite measure space and $\phi \in L^{\infty}(A)$. Prove that the multiplication operator M_{ϕ} has bounded inverse iff there exists $c > 0$ such that

$$|\phi(x)| \geq c$$

$(\forall x \in A \quad \text{a.e})$.

Solution: See Problem 67 in Halmos - A Hilbert space Problem Book.